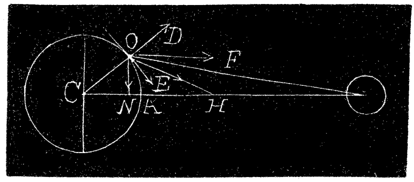
$$\therefore f'=\frac{ma\sin\varphi}{d^3}=Q\sin\varphi,$$

where $Q=ma/d^3$.



$$f=m\left(\frac{1}{NM^2}-\frac{1}{CM^2}\right)=m\left(\frac{CM^2-NM^2}{CM^2.NM^2}\right)$$

$$=\frac{mCN(CM+NM)}{CM^2.NM^2}=\frac{m.CN(2d-CN)}{d^2(d-CN)^2}$$

$$=\frac{2m.CN}{d^3}.\frac{(1-CN/2d)}{(1-CN/d)^2}.$$

Since CN/d is small we may neglect it

$$\therefore f=\frac{2m.CN}{d^3}=\frac{2ma\cos\varphi}{d^3}=2\,Q\cos\varphi.$$

Now let f=horizontal, f'=vertical components.

$$\therefore f=f\sin\varphi+f'\cos\varphi=3Q\sin\varphi\cos\varphi=\tfrac{3}{2}Q\sin2\varphi.$$

$$f'=f\cos\varphi-f'\sin\varphi=Q(2\cos^2\varphi-\sin^2\phi)=Q(3\cos^2\varphi-1).$$

DIOPHANTINE ANALYSIS.

68. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find a *general* value for p in the expression $4p+1$=the sum of two squares.

I. Solution by J. H. DRUMMOND, LL. D., Portland, Me., and G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Since $4p+1$ is odd one of the squares must be even and the other odd. Take $2q+1$ as one of the numbers and $2s$ for the other, and we have $4q^2+4q+1+4s^2=4p+1$. Hence $p=q^2+q+s^2$ in which q may be zero or any number; and s any number.

II. Solution by the PROPOSER.

From a table in which I have all the odd numbers, up to 12013, that are equivalent to the sum of two squares, I find by inspection that all the values for $4p+1$ can be obtained by making $p=\frac{n(n+1)}{2}+\frac{a(a+1)}{2}$, in which $n>a$.

Whence $4\left[\frac{n(n+1)}{2}+\frac{a(a+1)}{2}\right]+1=(n+a+1)^2+(n-a)^2$. (I).

When $a=0$, we have $4\left[\frac{n(n+1)}{2}\right]+1=(n+1)^2+n^2$; and we obtain the series of values 5, 13, 25, 41, 61, 85, etc.

When $a=1$, we have $4\left[\frac{n(n+1)}{2}+1\right]+1=(n+2)^2+(n-1)^2$; and we obtain the series of values 17, 29, 45, 65, 89, 117, etc.

When $a=2$, we have $4\left[\frac{n(n+1)}{2}+3\right]+1=(n+3)^2+(n-2)^2$; and we obtain the series of values 37, 53, 73, 97, 125, 157, etc.

We observe that the development of $4p+1$ occurs in *series*; the *number of terms in each series* as well as the *number of series* being infinite.

We also notice that the *number of series*$=a+1$. Now put $r=n-a$. We then readily deduce that the rth *term of any series*

$$=4\left[\frac{(r+a)(r+a+1)}{2}+\frac{a(a+1)}{2}\right]+1=(r+2a+1)^2+r^2. \quad \text{(II)}.$$

For the numerical development of $4p+1$, Formula II is better adapted than Formula I, as the values of r and a are independent of each other. But we can still improve a little in this direction. Put $N=a+1=$*the number of the series.*

Then (A), the rth term of the Nth series, and also, (B), the Nth term of the rth series

$$=4\left[\frac{(N+r)(N+r-1)}{2}+\frac{N(N-1)}{2}\right]+1=4\left[N(N+r-1)+\frac{r(r-1)}{2}\right]+1$$

$$=(2N+r-1)^2+r^2. \quad \text{(III)}.$$

From this we observe that there are *two forms of series* embodied in one formula, each form, however containing all the values of $4p+1$.

The one form, of which we have already treated and in which $r=$the consecutive integers for each value of N, or for each series, we shall designate as "Series A."

The other form, in which $N=$the consecutive integers for each value of r, or for each series, we shall term "Series B."

The *first series* of "Series B" consists of the *first terms* of the consecutive series of "Series A"; as 5, 17, 37, 65, 101, 145, etc.

The *second series* of "Series B" consists of the second terms of the consecutive series of "Series A"; as 13, 29, 53, 85, 125, 173, etc.; and so on for the respective series.

Two other values of p, in Formula III, are

$$\frac{(2N+r)(r-1)}{2}+N^2 \text{ and } (N-1)(N+r)+\frac{r(r+1)}{2},$$

obtained by different arrangement of terms and factoring. The two values are significant; as $4\left[\frac{(2N+r)(r-1)}{2}\right]$ is the difference between the rth and first terms in "Series A," and $4(N-1)(N+r)$ is the difference between the Nth and first terms of "Series B."

We have also a general formula for finding *consecutively* the terms of a series:—

In "Series A", rth term $+4(N+r)=r+1$th term.

In "Series B", Nth term $+4(2N+r)=N+1$th term.

The following mathematical diversions may be interesting as bearing upon this problem.

Knowing the first series in "Series A", we can find the other series consecutively by means of the following rule: *Add to each term of the last found series, omitting the first term, the number of the series times* 4, or $4N$; as

First series.	5,	13,	25,	41,	61,	85,	113,	etc.
Add 1×4		4,	4,	4,	4,	4,	4,	
Second series..................		17,	29,	45,	65,	89,	117,	etc.
Add 2×4			8,	8,	8,	8,	8,	
Third series......................			37,	53,	73,	97,	125,	etc., etc.

The number of the series$=N=\frac{1}{2}\sqrt{(1\text{st term}-1)}$; or $4N^2+1=$1st term.

In "Series B", $4\left[\frac{r(r+1)}{2}\right]+1=$1st term.

The following is a most interesting deduction from Formula III.

(1), $(2N+r-1)^2+r^2=(2N-1)(2N+2r-1)+2r^2=2r(2N+r-1)+(2N-1)^2$,

(2), $[(2N-1)(2N+2r-1)]^2+[2r(2N+r-1)]^2=[(2N+r-1)^2+r^2]^2$, or = the *square of* $4p+1=$the sum of two squares.

The application of these deductions gives rise to the annexed table, which I have constructed and extended to over three thousand numbers each equal to $4p+1=$the sum of two squares. It contains every number of the kind up to $12013=77^2+78^2$. By means of the table we can readily find—

(1). The two numbers the sum of whose squares=the given numerical value of $4p+1$.

(2). The two numbers the sum of whose squares=the square of the given numerical value of $4p+1$.

The *first column* consists of the consecutive values of r, and is called the "r column" of consecutive integers.

The other columns are the consecutive series of "Series A."

The *first row* is the "N row" of consecutive integers.

The *second row* is the "$2N-1$ row" of consecutive odd numbers.

The other rows are the consecutive series of "Series B."

The number of the column of values of $4p+1$ is indicated at its top by the value of N; and the number of the row of values is shown at its left by the value of r.

In using the table, all mention of values of r, N, and $2N-1$, refer to the respective values in the *same row* and the *same column* in which is found the given value of $4p+1$.

To find the two numbers the sum of whose squares$=4p+1$; one of the numbers$=r+2N-1$ and the other number$=r$. Take 97, at the intersection

of *column* 3 with *row* 4. Then $4+5=9$=one number, and 4=the other number; $97=9^2+4^2$.

To find the two numbers the sum of whose squares=the square of $4p+1$; one of the numbers$=4p+1-2r^2$, and the other number$=4p+1-(2N-1)^2$.

Take 97. Then $97-2\times4^2=65$=one of the numbers, and $97-5^2=72$; $97^2=65^2+72^2$.

TABLE OF VALUES OF $4p+1$=THE SUM OF TWO SQUARES.

	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	3	5	7	9	11	13	15	17	19	21	23	25
1	5	17	37	65	101	145	197	257	325	401	485	577	677
2	13	29	53	85	125	173	229	293	365	445	533	629	733
3	25	45	73	109	153	205	265	333	409	493	585	685	793
4	41	65	97	137	185	241	305	377	457	545	641	745	857
5	61	89	125	169	221	281	349	425	509	601	701	809	925
6	85	117	157	205	261	325	397	477	565	661	765	877	997
7	113	149	193	245	305	373	749	533	625	725	833	949	1073
8	145	185	233	289	353	425	505	593	689	793	905	1025	1153
9	181	225	277	337	405	481	565	657	757	865	981	1105	1237
10	221	269	325	389	461	541	629	725	829	941	1061	1189	1325
11	265	317	377	445	521	605	697	797	905	1021	1145	1277	1417
12	313	369	433	505	585	673	769	873	985	1105	1233	1369	1513
13	365	425	493	569	653	745	845	953	1069	1193	1325	1465	1613
14	421	485	557	637	725	821	925	1037	1157	1285	1421	1565	1717
15	481	549	625	709	801	901	1009	1125	1249	1381	1521	1669	1825
16	545	617	697	785	881	985	1097	1217	1345	1481	1625	1777	1937
17	613	689	773	865	965	1073	1189	1313	1445	1585	1733	1889	2053
18	685	765	853	949	1053	1165	1285	1413	1549	1693	1845	2005	2173
19	761	845	937	1037	1145	1261	1385	1517	1657	1805	1961	2125	2297
20	841	929	1025	1129	1241	1361	1489	1625	1769	1921	2081	2249	2425
21	925	1017	1117	1225	1341	1465	1597	1737	1885	2041	2205	2377	2557
22	1013	1109	1213	1325	1445	1573	1709	1853	2005	2165	2333	2509	2693
23	1105	1203	1313	1429	1553	1685	1825	1973	2129	2293	2465	2645	2833
24	1201	1305	1417	1537	1665	1801	1945	2097	2257	2425	2601	2785	2977
25	1301	1409	1525	1649	1781	1927	2069	2225	2389	2561	2741	2929	3125
26	1405	1517	1637	1765	1904	2045	2197	2357	2525	2701	2885	3077	3277
27	1513	1629	1753	1885	2025	2173	2329	2493	2665	2845	3033	3229	3433
28	1625	1745	1873	2009	2153	2305	2465	2633	2809	2993	3185	3385	3593

MISCELLANEOUS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are poured in at the two cusps, the quantity of each being such as would fill a length of the tube equal to its axis a. If the fluids do not mix, find the distance x_1, x_2 of the upper levels of the fluids from the vertex measured along the cycloidal arc. [From *Procter's Geometry of the Cycloid.*]